AUTOMATIC DIFFERENTIATION: EXERCISE

Given the function $f(x) : \mathbb{R}^5 \mapsto \mathbb{R}$ with

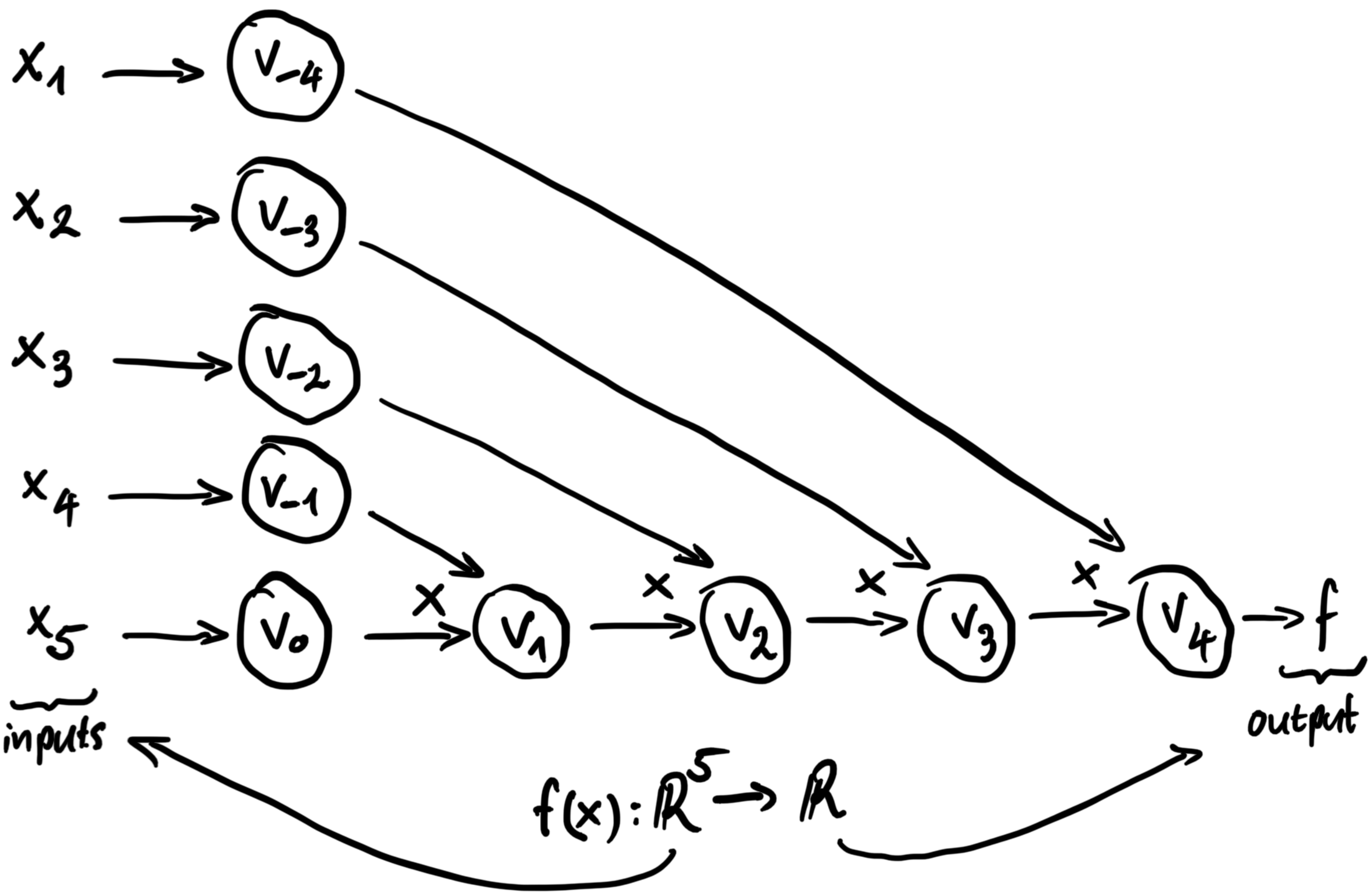
$$f(x) = x_1 x_2 x_3 x_4 x_5,$$

compute the gradient ∇f evaluated at the point $x = [2, 1, 1, 1, 1]^\intercal$.

1. Draw the computational graph.
2. Compute the gradient using forward mode. Note: you need $m = 5$ passes with different seed vectors. Write your solution in a evaluation table similar to what we did earlier.
3. Compute the gradient using reverse mode. Write your results in another evaluation table (with possibly fewer columns than forward mode above).
4. For both, forward and reverse mode, calculate the number of arithmetic operations (addition, subtraction, multiplication, division).

You may use the next two pages to write down your solution.
Work together with your neighbors.

AD: COMPUTATIONAL GRAPH



AD: FORWARD MODE

In the following the elements of the seed vectors are given by $p_i^{(j)} = \delta_{ij}$ with $i = 1, 2, \ldots, m$ and δ_{ij} the Kronecker delta.

Forward primal trace	Forward tangent trace	Pass with $p^{(1)}$	Pass with $p^{(2)}$	Pass with $p^{(3)}$	Pass with $p^{(4)}$	Pass with $p^{(5)}$
$v_{-4} = x_1 = 2$	$D_p v_{-4} = p_1^{(j)}$	$D_p v_{-4} = 1$	$D_p v_{-4} = 0$	$D_p v_{-4} = 0$	$D_p v_{-4} = 0$	$D_p v_{-4} = 0$
$v_{-3} = x_2 = 1$	$D_p v_{-3} = p_2^{(j)}$	$D_p v_{-3} = 0$	$D_p v_{-3} = 1$	$D_p v_{-3} = 0$	$D_p v_{-3} = 0$	$D_p v_{-3} = 0$
$v_{-2} = x_3 = 1$	$D_p v_{-2} = p_3^{(j)}$	$D_p v_{-2} = 0$	$D_p v_{-2} = 0$	$D_p v_{-2} = 1$	$D_p v_{-2} = 0$	$D_p v_{-2} = 0$
$v_{-1} = x_4 = 1$	$D_p v_{-1} = p_4^{(j)}$	$D_p v_{-1} = 0$	$D_p v_{-1} = 0$	$D_p v_{-1} = 0$	$D_p v_{-1} = 1$	$D_p v_{-1} = 0$
$v_0 = x_5 = 1$	$D_p v_0 = p_5^{(j)}$	$D_p v_0 = 0$	$D_p v_0 = 0$	$D_p v_0 = 0$	$D_p v_0 = 0$	$D_p v_0 = 1$
$v_1 = v_{-1} v_0 = 1$	$D_p v_1 = v_0 D_p v_{-1} + v_{-1} D_p v_0$	$D_p v_1 = 0$	$D_p v_1 = 0$	$D_p v_1 = 0$	$D_p v_1 = 1$	$D_p v_1 = 1$
$v_2 = v_{-2} v_1 = 1$	$D_p v_2 = v_1 D_p v_{-2} + v_{-2} D_p v_1$	$D_p v_2 = 0$	$D_p v_2 = 0$	$D_p v_2 = 1$	$D_p v_2 = 1$	$D_p v_2 = 1$
$v_3 = v_{-3} v_2 = 1$	$D_p v_3 = v_2 D_p v_{-3} + v_{-3} D_p v_2$	$D_p v_3 = 0$	$D_p v_3 = 1$	$D_p v_3 = 1$	$D_p v_3 = 1$	$D_p v_3 = 1$
$v_4 = v_{-4} v_3 = 2$	$D_p v_4 = v_3 D_p v_{-4} + v_{-4} D_p v_3$	$D_p v_4 = 1$	$D_p v_4 = 2$	$D_p v_4 = 2$	$D_p v_4 = 2$	$D_p v_4 = 2$

independent variables — *dependent* variables

AD: REVERSE MODE

Forward pass:

Intermediate	Partial Derivatives
$v_{-4} = x_1 = 2$	
$v_{-3} = x_2 = 1$	
$v_{-2} = x_3 = 1$	
$v_{-1} = x_4 = 1$	
$v_0 = x_5 = 1$	
$v_1 = v_{-1} v_0 = 1$	$\frac{\partial v_1}{\partial v_{-1}} = v_0 = 1\,;\ \frac{\partial v_1}{\partial v_0} = v_{-1} = 1$
$v_2 = v_{-2} v_1 = 1$	$\frac{\partial v_2}{\partial v_{-2}} = v_1 = 1\,;\ \frac{\partial v_2}{\partial v_1} = v_{-2} = 1$
$v_3 = v_{-3} v_2 = 1$	$\frac{\partial v_3}{\partial v_{-3}} = v_2 = 1\,;\ \frac{\partial v_3}{\partial v_2} = v_{-3} = 1$
$v_4 = v_{-4} v_3 = 2$	$\frac{\partial v_4}{\partial v_{-4}} = v_3 = 1\,;\ \frac{\partial v_4}{\partial v_3} = v_{-4} = 2$

independent variables | *dependent* variables

Reverse pass:

Adjoint
$\bar{v}_{-4} = \frac{\partial f}{\partial v_4}\frac{\partial v_4}{\partial v_{-4}} = \bar{v}_4 \frac{\partial v_4}{\partial v_{-4}} = 1$
$\bar{v}_{-3} = \frac{\partial f}{\partial v_3}\frac{\partial v_3}{\partial v_{-3}} = \bar{v}_3 \frac{\partial v_3}{\partial v_{-3}} = 2$
$\bar{v}_{-2} = \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial v_{-2}} = \bar{v}_2 \frac{\partial v_2}{\partial v_{-2}} = 2$
$\bar{v}_{-1} = \frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial v_{-1}} = \bar{v}_1 \frac{\partial v_1}{\partial v_{-1}} = 2$
$\bar{v}_0 = \frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial v_0} = \bar{v}_1 \frac{\partial v_1}{\partial v_0} = 2$
$\bar{v}_1 = \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial v_1} = \bar{v}_2 \frac{\partial v_2}{\partial v_1} = 2$
$\bar{v}_2 = \frac{\partial f}{\partial v_3}\frac{\partial v_3}{\partial v_2} = \bar{v}_3 \frac{\partial v_3}{\partial v_2} = 2$
$\bar{v}_3 = \frac{\partial f}{\partial v_4}\frac{\partial v_4}{\partial v_3} = \bar{v}_4 \frac{\partial v_4}{\partial v_3} = 2$
$\bar{v}_4 = \frac{\partial f}{\partial v_4} = \frac{\partial v_4}{\partial v_4} = 1$

AD: COMPUTATIONAL COMPLEXITY

- ***Forward mode:***
  - Computation of v_j for $j = 1, 2, 3, 4$: 4mul
  - Computation of $D_p v_j$ for $j = 1, 2, 3, 4$: $4 \times (2\text{mul} + 1\text{add}) = 12\text{ops}$
  - Computation of $D_p v_j$ depends on $j = 1, 2, \ldots, m$ passes with $p^{(j)}$.
  - *Total:* $4\text{mul} + m \times 12\text{ops} = 4 + 5 \times 12 = 64\text{ops}$
- ***Reverse mode:***
  - Computation of v_j for $j = 1, 2, 3, 4$: 4mul
  - Reverse pass: 8mul
  - *Total:* $4\text{mul} + 8\text{mul} = 12\text{ops}$